[T.K] 3.8.5.1 Let $(\mathcal{F}_n)_{n\geq 1}$ be a filtration and X a random variable; $\mathbb{E}[|X|] < \infty$.
Set $X_n = \mathbb{E}[X|\mathcal{F}_n]$.

To show : $(X_n)_{n\geq 1}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 1}$.

Definitions: * Let Ω be some space. $(\mathcal{F}_n)_{n\geq 1}$ is called a filtration if $\mathcal{F}_n$ is a σ-algebra on Ω $\forall n\geq 1$ and $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ $\forall n\geq 1$.

* Let $(\mathcal{F}_n)_{n\geq 1}$ be a filtration and for each $n\geq 1$, let $Z_n : \Omega \to \mathbb{R}$ be a $\mathcal{F}_n$-measurable random variable.
The sequence $(Z_n)_{n\geq 1}$ is a martingale if $\mathbb{E}[Z_n] < \infty$ $\forall n\geq 1$ and $\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = Z_n$ $\forall n\geq 1$.

We just need to check these properties.

1./ By definition of conditional expectation, $X_n = \mathbb{E}[X|\mathcal{F}_n]$ is $\mathcal{F}_n$-measurable.

2./ $$\mathbb{E}[X_n] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_n]] \underset{\text{property of conditional expectation}}{=} \mathbb{E}[X] < \infty, \quad \forall n\geq 1.$$

3./ $$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_{n+1}]|\mathcal{F}_n] \overset{\text{Tower property}}{=} \mathbb{E}[X|\mathcal{F}_n] = X_n$$

$\Rightarrow (X_n)_{n\geq 1}$ is a martingale!

QED.

[T.K] 3.8.5.3 Let $(X_n)_{n\geq 1}$ be a martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$.

To show : $\forall n\geq 1$, $\mathbb{E}[X_{n+1}] = \mathbb{E}[X_n] = \dots = \mathbb{E}[X_1]$.

By induction : $\mathbb{E}[X_1] = \mathbb{E}[X_1]$: ok

Suppose $\mathbb{E}[X_k] = \mathbb{E}[X_1]$ for some $k\geq 1$, we want to see that $\mathbb{E}[X_{k+1}] = \mathbb{E}[X_1]$.

$$\mathbb{E}[X_{k+1}] \underset{\substack{\text{property of}\\ \text{conditional expectation}}}{=} \mathbb{E}[\mathbb{E}[X_{k+1}|\mathcal{F}_k]] \underset{(X_n)_{n\geq 1} \text{ is a martingale}}{=} \mathbb{E}[X_k] \overset{\text{induction's assumption.}}{=} \mathbb{E}[X_1]$$

(You can see it as a consequence of the)
Tower property

Hence $\forall n\geq 1$, $\mathbb{E}[X_n] = \mathbb{E}[X_1]$.

QED

[T.K] 3.8.5.4 Let $(X_n)_{n\geq 1}$ be a martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$.

To show: $\forall n\geq m\geq 1$, $\mathbb{E}[X_n|\mathcal{F}_m] = X_m$

Fix $m\geq 1$ and show the result by induction on $n\geq m$.

if $n=m$, $\mathbb{E}[X_m|\mathcal{F}_m] = X_m$, because X_m is $\mathcal{F}_m$-measurable since $(X_n)_{n\geq 1}$ is a martingale.

Suppose $\mathbb{E}[X_{m+k}|\mathcal{F}_m] = X_m$ for some $k\geq 0$. We want to see that $\mathbb{E}[X_{m+k+1}|\mathcal{F}_m] = X_m$.

$$\mathbb{E}[X_{m+k+1}|\mathcal{F}_m] \underset{\text{Tower property}}{=} \mathbb{E}[\mathbb{E}[X_{m+k+1}|\mathcal{F}_{m+k}]|\mathcal{F}_m] \underset{(X_n)_{n\geq 1}\text{ is a martingale}}{=} \mathbb{E}[X_{m+k}|\mathcal{F}_m] \overset{\text{induction's assumption}}{=} X_m$$

Thus $\forall n\geq m\geq 1$, $\mathbb{E}[X_n|\mathcal{F}_m] = X_m$.

QED

[T.K] 3.8.5.5 Let $(X_n)_{n\geq 1}$ be an independent an identically distributed sequence of random variables with $\mathbb{E}[X_n]=\mu$ and $Var[X_n]=\sigma^2$ $\forall n\geq 1$.

Define $W_0=0$ and $W_n=\sum_{i=1}^{n} X_i$ $\forall n\geq 1$. Also set $\mathcal{F}_n := \sigma(X_1, X_2, \dots, X_n)$ $\forall n\geq 1$ and $S_n = (W_n - n\mu)^2 - n\sigma^2$

To show: $(S_n)_{n\geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$.

1/ We want to show that $\forall n\geq 0$, S_n is $\mathcal{F}_n$-measurable.

S_n can be seen as the following map: $(X_1, X_2, \dots, X_n) \overset{f_1}{\longmapsto} \sum_{i=1}^{n} X_i \overset{f_2}{\longmapsto} \sum_{i=1}^{n} X_i - n\mu \overset{f_3}{\longmapsto} (\sum_{i=1}^{n} X_i - n\mu)^2 \overset{f_4}{\longmapsto} (\sum_{i=1}^{n} X_i - n\mu)^2 - n\sigma^2$

$f_1 : \mathbb{R}^n \to \mathbb{R}$ is continuous
$f_2, f_3, f_4 : \mathbb{R}\to\mathbb{R}$ are continuous.

By the course, all continuous functions preserve the measurability on $\mathbb{R}$, i.e. $f^{-1}((-\infty, a])$ is measurable in $\mathbb{R}$, $\forall a\in\mathbb{R}$.

By construction of $\sigma(X_1, X_2, \dots, X_n)$, $(X_1, X_2, \dots, X_n)$ is measurable with respect to $\sigma(X_1, X_2, \dots, X_n)$

Hence $\forall n\geq 1$, $S_n = f_4\circ f_3\circ f_2\circ f_1 (X_1, X_2, \dots, X_n)$ is still measurable with respect to $\sigma(X_1, X_2, \dots, X_n)$.

OK!

2/ We want to show that $\mathbb{E}[S_n]<\infty$ $\forall n\geq 0$. $\mathbb{E}[S_0] = \mathbb{E}[W_0^2] = 0$.

$$\mathbb{E}[S_n] = \mathbb{E}[(W_n - n\mu)^2 - n\sigma^2] \underset{\text{linearity of }\mathbb{E}}{=} \mathbb{E}[(W_n-n\mu)^2] - n\sigma^2 \underset{\text{develop the square}}{=} \mathbb{E}[W_n^2 - 2n\mu W_n + n^2\mu^2] - n\sigma^2 \overset{\text{linearity of }\mathbb{E}}{=} \mathbb{E}[W_n^2] - 2n\mu\,\mathbb{E}[W_n] + n^2\mu^2 - n\sigma^2$$

$$\text{Now } \mathbb{E}[W_n] = \mathbb{E}[\sum_{i=1}^{n} X_i] \overset{\text{linearity of }\mathbb{E}}{=} \sum_{i=1}^{n}\mathbb{E}[X_i] \overset{\mathbb{E}[X_i]=\mu\ \forall i\geq 1}{=} n\mu \quad\text{and}\quad \mathbb{E}[W_n^2] = \mathbb{E}[(\sum_{i=1}^{n} X_i)^2] = \mathbb{E}[\sum_{i=1}^{n}\sum_{j=1}^{n} X_iX_j] = \sum_{i\neq j}\underbrace{\mathbb{E}[X_iX_j]}_{=\mathbb{E}[X_i]\mathbb{E}[X_j]=\mu^2 \text{ by independence}} + \sum_{i=1}^{n}\mathbb{E}[X_i^2]$$

And $\mathbb{E}[X_i^2] = Var[X_i] + \mathbb{E}[X_i]^2 = \sigma^2 + \mu^2$

By definition of the variance; $Var[X_i] = \sigma^2 \ \forall i \geq 1$ and $\mathbb{E}[X_i] = \mu \ \forall i \geq 1$

Hence $\mathbb{E}[W_n^2] = \sum_{i \neq j} \underbrace{\mathbb{E}[X_i X_j]}_{= \mathbb{E}[X_i]\mathbb{E}[X_j] = \mu^2 \text{ independence}} + \sum_{i=1}^{n} \mathbb{E}[X_i^2] = \sum_{i \neq j} \mu^2 + \sum_{i=1}^{n} (\mu^2 + \sigma^2) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mu^2 + \sum_{i=1}^{n} \sigma^2 = n^2\mu^2 + n\sigma^2$

Thus $\mathbb{E}[S_n] = \mathbb{E}[W_n^2] - 2n\mu\,\mathbb{E}[W_n] + n^2\mu^2 - n\sigma^2 = n^2\mu^2 + n\sigma^2 - 2n\mu(n\mu) + n^2\mu^2 - n\sigma^2 = 0 < \infty$

Consequently $\mathbb{E}[S_n] < \infty \ \forall n \geq 1$.

ok!

3.) We want to see that $\mathbb{E}[S_{n+1} | \mathcal{F}_n] = S_n \ \forall n \geq 1$.

First remark that since $S_n = \mathbb{E}[S_n | \mathcal{F}_n]$ (S_n is $\mathcal{F}_n$-measurable), $\mathbb{E}[S_{n+1}|\mathcal{F}_n] = S_n \Leftrightarrow \mathbb{E}[S_{n+1} - S_n | \mathcal{F}_n] = 0 \quad \forall n \geq 1$.

Now $\mathbb{E}[S_{n+1} - S_n | \mathcal{F}_n] = \mathbb{E}[\underbrace{((W_{n+1} - (n+1)\mu)^2 - (n+1)\sigma^2) - ((W_n - n\mu)^2 - n\sigma^2)}_{=: A} | \mathcal{F}_n]$.

Let's develop A: $(W_{n+1} - (n+1)\mu)^2 - (n+1)\sigma^2 = W_{n+1}^2 - 2(n+1)\mu W_{n+1} + (n+1)^2\mu^2 - (n+1)\sigma^2$

$(W_n - n\mu)^2 - n\sigma^2 = W_n^2 - 2n\mu W_n + n^2\mu^2 - n\sigma^2$

Hence $A = (W_{n+1}^2 - W_n^2) - 2n\mu \underbrace{(W_{n+1} - W_n)}_{= \sum_{i=1}^{n+1} X_i - \sum_{i=1}^{n} X_i = X_{n+1}} - 2\mu W_{n+1} + \mu^2 \underbrace{((n+1)^2 - n^2)}_{= n^2 + 2n + 1 - n^2 = 2n+1} - \sigma^2 \underbrace{((n+1) - n)}_{=1}$

$\Rightarrow \mathbb{E}[A|\mathcal{F}_n] \overset{\text{linearity of } \mathbb{E}[\cdot|\mathcal{F}_n]}{=} \underbrace{\mathbb{E}[W_{n+1}^2 - W_n^2 | \mathcal{F}_n]}_{①} - 2n\mu \underbrace{\mathbb{E}[X_{n+1}|\mathcal{F}_n]}_{②} - 2\mu \underbrace{\mathbb{E}[W_{n+1}|\mathcal{F}_n]}_{③} + (2n+1)\mu^2 - \sigma^2$

① $\mathbb{E}[W_{n+1}^2 - W_n^2 | \mathcal{F}_n] = \mathbb{E}[W_{n+1}^2 | \mathcal{F}_n] - W_n^2 = \mathbb{E}[(W_n + X_{n+1})^2 | \mathcal{F}_n] - W_n^2 = \mathbb{E}[W_n^2 + 2W_n X_{n+1} + X_{n+1}^2 | \mathcal{F}_n] - W_n^2$

(W_n^2 is $\mathcal{F}_n$-measurable; by definition of W_{n+1})

$= \underbrace{\mathbb{E}[W_n^2 | \mathcal{F}_n]}_{= W_n^2} + 2\underbrace{\mathbb{E}[W_n \cdot X_{n+1} | \mathcal{F}_n]}_{= 2W_n \cdot \mathbb{E}[X_{n+1}|\mathcal{F}_n]} + \underbrace{\mathbb{E}[X_{n+1}^2 | \mathcal{F}_n]}_{= \mathbb{E}[X_{n+1}^2] = \sigma^2 + \mu^2} - W_n^2 = W_n^2 + 2W_n\mu + \sigma^2 + \mu^2 - W_n^2 = 2W_n\mu + \sigma^2 + \mu^2$

$\hookrightarrow X_{n+1}^2$ is independent of $X_1, X_2, \ldots, X_n$.

W_n^2 is $\mathcal{F}_n$-measurable

$= 2W_n \cdot \mathbb{E}[X_{n+1}] = 2W_n \cdot \mu$

$\hookrightarrow X_{n+1}$ is independent of $X_1, X_2, \ldots, X_n$.

W_n is $\mathcal{F}_n$-measurable

②
$$\mathbb{E}[X_{n+1} / \mathcal{F}_n] \underset{/}{=} \mathbb{E}[X_{n+1}] = \mu.$$

X_{n+1} is independent of
$X_1, X_2, \ldots, X_n$

linearity of $\mathbb{E}[\cdot / \mathcal{F}_n]$

③ $\mathbb{E}[W_{n+1} / \mathcal{F}_n] \overset{/}{=} \sum_{i=1}^{n+1} \mathbb{E}[X_i / \mathcal{F}_n] = \sum_{i=1}^{n} \mathbb{E}[X_i / \mathcal{F}_n] + \mathbb{E}[X_{n+1} / \mathcal{F}_n] \underset{/}{=} \sum_{i=1}^{n} X_i + \mathbb{E}[X_{n+1}] = W_n + \mu.$

X_i is $\mathcal{F}_n$-measurable $\forall 1 \le i \le n$
X_{n+1} is independent of $X_1, X_2, \ldots, X_n$

Hence $\mathbb{E}[S_{n+1} - S_n / \mathcal{F}_n] = \mathbb{E}[A / \mathcal{F}_n] = 2W_n\mu + \sigma^2 + \mu^2 - 2n\mu^2 - 2\mu(W_n + \mu) + (2n+1)\mu^2 - \sigma^2$

$= 2W_n\mu + \sigma^2 + \mu^2 - 2n\mu^2 - 2W_n\mu - 2\mu^2 + 2n\mu^2 + \mu^2 - \sigma^2 = 0$ OK!

Hence $(S_n)_{n \ge 0}$ is a martingale.

QED

Remark: Part 2. could be skipped if each S_n is bounded either from above or from below. Then if you show 3./, compute $\mathbb{E}[S_0]$ and evoque [T.K] 3.8.5.3, $(S_n)_{n \ge 1}$ being a martingale is proved.

[T.K] 3.8.5.6 (a). Let $(X_n)_{n \ge 1}$ be a sequence of independent random variables.
Let f and g be two distinct probability densities on $\mathbb{R}$.
Set $L_n := \dfrac{f(X_0) \cdot f(X_1) \cdot \ldots \cdot f(X_n)}{g(X_0) \cdot g(X_1) \cdot \ldots \cdot g(X_n)}$ $\forall n \ge 1$ and assume that $g(x) > 0$ $\forall x \in \mathbb{R}$.

To show: $(L_n)_{n \ge 1}$ is a martingale with respect to $\sigma(X_0, X_1, \ldots, X_n)$, if $\forall H$ Borel, $\mathbb{E}[H(X)] = \int_{-\infty}^{+\infty} H(x) g(x) dx$

1./ L_n is $\sigma(X_0, X_1, \ldots, X_n)$-measurable $\forall n \ge 1$.
Since f and g are probability densities on $\mathbb{R}$, they are measurable on $\mathbb{R}$ and since $L_n: (X_0, X_1, \ldots, X_n) \overset{f_1}{\longmapsto} \frac{f(X_0)}{g(X_0)} \cdot \ldots \cdot \frac{f(X_n)}{g(X_n)}$
f_1 is measurable on $\mathbb{R}$ and L_n is $\sigma(X_0, X_1, \ldots, X_n)$-measurable.

2./ $\mathbb{E}[L_0] = \mathbb{E}\left[\frac{f(X_0)}{g(X_0)}\right] = \int_{-\infty}^{+\infty} \frac{f(x)}{g(x)} g(x) dx = \int_{-\infty}^{+\infty} f(x) dx = 1$ since f is a probability density.
Hence $\mathbb{E}[L_0] < \infty$.
By the previous remark, if we can show 3./, the sequence will be a martingale, since $L_n > 0$ $\forall n \ge 1$.

3./ We want to see that $\mathbb{E}[L_{n+1} - L_n / \mathcal{F}_n] = 0$ $\forall n \ge 1$. Fix $n \ge 0$.

$\mathbb{E}[L_{n+1} - L_n / \mathcal{F}_n] = \mathbb{E}\left[\frac{f(X_0)}{g(X_0)} \cdot \ldots \cdot \frac{f(X_n)}{g(X_n)} \cdot \frac{f(X_{n+1})}{g(X_{n+1})} - \frac{f(X_0)}{g(X_0)} \cdot \ldots \cdot \frac{f(X_n)}{g(X_n)} / \mathcal{F}_n\right] = \mathbb{E}\left[\left(\frac{f(X_0)}{g(X_0)} \cdot \ldots \cdot \frac{f(X_n)}{g(X_n)}\right)\left(\frac{f(X_{n+1})}{g(X_{n+1})} - 1\right) / \mathcal{F}_n\right]$

$\underset{/}{=} \frac{f(X_0)}{g(X_0)} \cdot \ldots \cdot \frac{f(X_n)}{g(X_n)} \cdot \mathbb{E}\left[\frac{f(X_{n+1})}{g(X_{n+1})} - 1 / \mathcal{F}_n\right] \underset{/}{=} \frac{f(X_0)}{g(X_0)} \cdot \ldots \cdot \frac{f(X_n)}{g(X_n)} \cdot \mathbb{E}\left[\frac{f(X_{n+1})}{g(X_{n+1})} - 1\right] \underset{/}{=} \frac{f(X_0)}{g(X_0)} \cdot \ldots \cdot \frac{f(X_n)}{g(X_n)} \cdot \Big(\underbrace{\int_{-\infty}^{+\infty} f(x) dx - 1}_{=0}\Big) = 0.$

$\frac{f(X_0)}{g(X_0)} \cdot \ldots \cdot \frac{f(X_n)}{g(X_n)}$ is $\mathcal{F}_n$-measurable

$\frac{f(X_{n+1})}{g(X_{n+1})}$ is independent of $X_1, X_2, \ldots, X_n$

by how $\mathbb{E}$ is defined in this problem.

$\Rightarrow (L_n)_{n \ge 1}$ is a martingale.

QED